

ON SELF-SIMILARITY OF WREATH PRODUCTS OF ABELIAN GROUPS

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ABSTRACT. We prove that a self-similar free abelian group has finite rank. We apply the result to self-similar wreath products of abelian groups $G = BwrX$. We show that if X is torsion-free, then B is torsion of finite exponent. Furthermore, we construct a self-similar group $G = BwrC_2$ where B is free abelian of infinite rank.

1. INTRODUCTION

A group G is self-similar provided for some finite positive integer m , the group has a faithful representation on an infinite regular one-rooted m -tree \mathcal{T}_m such that the representation is state-closed and is transitive on the tree's first level. If a group G does not admit such a representation for any m then we say G is not self-similar.

Another equivalent formulation of self similarity is via virtual endomorphisms: let G be a group with a subgroup H of finite index m . A homomorphism $f : H \rightarrow G$ is called a *virtual endomorphism* of G and (H, f) is called a similarity pair for G . This setup produces a homomorphism φ of G into the group of automorphisms of \mathcal{T}_m whose kernel K is the largest subgroup of H which is both normal in G and f -invariant (in the sense $K^f \leq K$). If this kernel is trivial then G is self-similar and f is said to be *simple*.

Which groups admit faithful self-similar representation is an on going topic of investigation. The first in depth study of this question was undertaken in [2] and then in book form in [3]. Faithful self-similar representations are known for many individual finitely generated groups ranging from the torsion groups of Grigorchuk and Gupta-Sidki to free groups. Such representations have been also studied for the family of abelian groups [4], of finitely generated nilpotent groups [8], as well as for arithmetic groups [10]. One class which has received attention in recent years is that of wreath products of abelian groups $G = BwrX$, such as the classical lamplighter group [1] in which B is cyclic of order 2 and X is infinite cyclic; see [9] for further references.

A question which was raised in the end of [4] concerns *the existence of self-similar free abelian groups of infinite rank*. Using results from the same paper, we provide a negative answer.

Recalling that the *free rank* of an abelian group A is the maximum of the ranks of free abelian subgroups of A , we prove more generally

Theorem 1. *The free rank of a self-similar abelian group is finite.*

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This result complements a previous one to the effect that *a torsion self-similar abelian group can have infinite rank yet it must have finite exponent* ([4], Theorem 2, part (i)).

It is interesting and suggestive to view Theorem 1 as a result on the existence of invariant subgroups for virtual endomorphisms. Specifially,

Corollary 1. *Any virtual endomorphism of a free abelian group of infinite rank leaves invariant some nontrivial subgroup.*

In the realm of nilpotent groups, we note that those which are finitely generated and torsion-free of nilpotency class at most 2 are self-similar ([8], Theorem 9). However, for larger classes, Mathieu Olivier has announced recently the construction of finitely generated torsion-free nilpotent groups which are not self-similar, using graded nilpotent Lie algebras [7].

We apply Theorem 1 to metabelian groups of wreath type

Theorem 2. *Let $G = BwrX$ be a self-similar wreath product of abelian groups. If X is torsion free then B is a torsion group of finite exponent. In particular, $\mathbb{Z}wr\mathbb{Z}$ cannot be self-similar.*

Observe that the case $G = C_pwrX$ where C_p is cyclic of prime order p and X is free abelian of rank $d \geq 1$ was the subject of [9]. In it, self-similar groups of this type are constructed for every finite rank d . Moreover, observe that though $G = \mathbb{Z}wr\mathbb{Z}$ is not self-similar, it has a faithful finite-state representation on the binary tree [5].

We finish this paper by constructing a self-similar group $G = BwrC_2$ where B is free abelian of infinite rank. Thus, in spite of the fact that the normal closure of B in G is free abelian of infinite rank and therefore not self-similar, still it embeds as a subgroup of index 2 in the self-similar group G .

2. PRELIMINARIES

We recall a number of notions of groups acting on trees and of virtual endomorphisms from [4].

2.1. State-closed groups. 1. Automorphisms of one-rooted regular trees $\mathcal{T}(Y)$ indexed by finite sequences from a finite set Y of size $m \geq 2$, have a natural interpretation as automata on the alphabet Y , and with states which are again automorphisms of the tree. A subgroup G of the group of automorphisms $\mathcal{A}(Y)$ of the tree is said to have degree m . Moreover, G is *state-closed* (in the language of automata) of degree m provided the states of its elements are themselves elements of the same group.

2. An automorphism group G of the tree group is said to be *transitive* provided the permutation group $P(G)$ induced by G on the set Y is transitive.

3. A group G is said to be *self-similar* provided it is a state-closed and transitive subgroup of $\mathcal{A}(Y)$ for some finite set Y .

4. The automorphism group $\mathcal{A}(Y)$ of the tree is a topological group with respect to the topology inherited from the tree. This topology allows us to exponentiate elements of $\mathcal{A}(Y)$ by m -ary integers from \mathbb{Z}_m . Given a subgroup G of $\mathcal{A}(Y)$, its topological closure \overline{G} with respect to the tree topology belongs to the same variety as G . Also, if G is state-closed then so is \overline{G} .

5. The diagonal map $\alpha \rightarrow \alpha^{(1)} = (\alpha, \alpha, \dots, \alpha)$, an m -tuple, is a monomorphism of \mathcal{A}_m . Define inductively $\alpha^{(0)} = \alpha$, $\alpha^{(i+1)} = (\alpha^{(i)})^{(1)}$ for $i \geq 0$. It is convenient to introduce a symbol x and write $\alpha^{(i)}$ as α^{x^i} for $i \geq 0$. This permits more general exponentiation by formal power series $p(x) \in \mathbb{Z}_m[[x]]$. Given a subgroup G of $\mathcal{A}(Y)$, its *diagonal closure* is the group $\tilde{G} = \langle G^{(i)} \mid i \geq 0 \rangle$. Observe that the diagonal closure operation preserves the state-closed property.

6. Given an abelian transitive state-closed group A , its diagonal closure \tilde{A} is again abelian. The composition of the diagonal and topological closures when applied to A produces an abelian group denoted by A^* which can be viewed additively as a finitely generated $\mathbb{Z}_m[[x]]$ -module.

2.2. Virtual endomorphisms. We elaborate on the quick definitions made in the introduction.

1. Let G be a group with a subgroup H of finite index m . A homomorphism $f : H \rightarrow G$ is called a *virtual endomorphism* of G and (G, H, f) is called a *similarity triple*; if G is fixed then (H, f) is called a *similarity pair*.

2. Let G be a transitive state-closed subgroup of $\mathcal{A}(Y)$ where $Y = \{1, 2, \dots, m\}$ and denote the subgroup of $\mathcal{A}(Y)$ which fixes the vertex 1 by $\text{Fix}_G(1)$. Then $[G : \text{Fix}_G(1)] = m$ and the projection on the 1st coordinate of $\text{Fix}_G(1)$ produces a subgroup of G ; that is, $\pi_1 : \text{Fix}_G(1) \rightarrow G$ is a virtual endomorphism of G .

3. Let G be a group with a subgroup H of finite index m and a homomorphism $f : H \rightarrow G$. If $U \leq H$ and $U^f \leq U$ then U is called *f-invariant*. The largest subgroup K of H , which is normal in G and is *f-invariant* is called the *f-core*(H). If the *f-core*(H) is trivial then f and the triple (G, H, f) are called *simple*.

4. Given a triple (G, H, f) and a right transversal $L = \{x_1, x_2, \dots, x_m\}$ of H in G , the permutational representation $\pi : G \rightarrow \text{Perm}(1, 2, \dots, m)$ is $g^\pi : i \rightarrow j$ which is induced from the right multiplication $Hx_i g = Hx_j$. Generalizing the Kalujnine-Krasner procedure [6], we produce recursively a representation $\varphi : G \rightarrow \mathcal{A}(m)$, defined by

$$g^\varphi = \left(\left(x_i g \cdot (x_{(i)g^\pi})^{-1} \right)^{f^\varphi} \right)_{1 \leq i \leq m} g^\pi,$$

seen as an element of an infinitely iterated wreath product of $\text{Perm}(1, 2, \dots, m)$. The kernel of φ is precisely the *f-core*(H) and G^φ is state-closed and transitive and $H^\varphi = \text{Fix}_{G^\varphi}(1)$.

Lemma 1. *A group G is self-similar if and only if there exists a simple similarity pair (H, f) for G .*

3. PROOF OF THEOREM 1

We will use from [4] the following two fundamental results.

Theorem 3. *Let A be an abelian transitive state-closed group of degree m . Then, (1) the group A^* is isomorphic to a finitely presented $\mathbb{Z}_m[[x]]$ -module; (2) if A^* is torsion-free then it is a finitely generated \mathbb{Z}_m -module which is also a pro- m group.*

Theorem 4. *Let A be an abelian transitive state-closed group of degree m and $\text{tor}(A)$ its torsion subgroup. Then, (i) $\text{tor}(A)$ is a direct summand of A and has exponent a divisor of the exponent of $P(A)$; (ii) the action of A on the m -ary tree induces transitive state-closed representations of $\text{tor}(A)$ on the m_1 -tree and*

of $\frac{A}{\text{tor}(A)}$ on the m_2 -tree, where $m_1 = |P(\text{tor}(A))|$ and $m_2 = |\frac{P(A)}{P(\text{tor}(A))}|$; (iii) if $A = \text{tor}(A)$ and $P(A) \cong \oplus_{1 \leq i \leq k} \frac{\mathbb{Z}}{n_i \mathbb{Z}}$, then $A^* \cong \oplus_{1 \leq i \leq k} \frac{\mathbb{Z}}{n_i \mathbb{Z}}[[x]]$.

We finish the proof of Theorem 1.

Proof. Let m be a minimum degree of a tree \mathcal{T}_m for which there exists a subgroup A of $\text{Aut}(\mathcal{T}_m)$ which is a counterexample to the assertion. We may assume A to be diagonally and topologically closed. If the torsion subgroup $\text{tor}(A)$ is nontrivial then the quotient group $\bar{A} = \frac{A}{\text{tor}(A)}$ has a faithful state-closed and transitive representation on a tree of degree smaller than m (Theorem 5, part (ii)) while the free rank of \bar{A} continues to be infinite, which is absurd. Therefore, $\text{tor}(A)$ is trivial and the m -congruence property applies (Theorem 4, part (2)); that is,

$$S = \text{Stab}_A(l) \leq A^m \leq A$$

for some finite l . However, as $\frac{A}{S}$ is finite, it follows that $\frac{A}{A^m}$ is finite and therefore the free rank of A is finite and we have reached a contradiction. \square

4. PROOF OF THEOREM 2

We recall B, X are abelian groups, X is a torsion-free group and $G = BwrX$. Denote the normal closure of B in G by $A = B^G$. Let (H, f) be the similarity pair with respect to which G is self-similar and let $[G : H] = m$. Define

$$A_0 = A \cap H, \quad L = (A_0)^f \cap A, \quad Y = X \cap (AH).$$

Note that if $x \in X$ is nontrivial then the centralizer $C_A(x)$ is trivial. We develop the proof in four lemmas.

Lemma 2. *Either B^m is trivial or $(A_0)^f \leq A$. In both cases $A \neq A_0$.*

Proof. We have $A^m \leq A_0$ and $X^m \leq H$. Then,

$$\begin{aligned} [A^m, X^m] &\triangleleft G, \\ [A^m, X^m] &\leq [A_0, X^m] \leq A_0. \end{aligned}$$

Also,

$$\begin{aligned} f : [A^m, X^m] &\rightarrow [(A^m)^f, (X^m)^f] \leq (A_0)^f \cap G' \\ &\leq (A_0)^f \cap A = L. \end{aligned}$$

(1) If L is trivial then $[A^m, X^m] \leq \ker(f)$. Since f is simple, it follows that $\ker(f) = 1$ and $[A^m, X^m] = 1 = [B^m, X^m]$. As $X^m \neq 1$, we conclude $A^m = 1 = B^m$. (2) If L is nontrivial then L is central in $M = A(A_0)^f = A(X \cap M)$ which implies $X \cap M = 1$ and $(A_0)^f \leq A$. (3) If B is a torsion group then $\text{tor}(G) = A$; clearly, $(A_0)^f \leq A$ and $A \neq A_0$. \square

Let G be a counterexample; that is, B has infinite exponent. By the previous lemma $(A_0)^f \leq A$ and so we may use Proposition 1 of [9] to replace the simple similarity pair (H, f) by a simple pair (\dot{H}, \dot{f}) where $\dot{H} = A_0 Y$ ($Y \leq X$) and $(Y)^{\dot{f}} \leq X$. In other words, we may assume $(Y)^f \leq X$.

Lemma 3. *If $z \in X$ is nontrivial and $x_1, \dots, x_t, z_1, \dots, z_l \in X$, then there exists an integer k such that*

$$z^k \{z_1, \dots, z_l\} \cap \{x_1, \dots, x_t\} = \emptyset.$$

Proof. Note that the set $\{k \in \mathbb{Z} \mid \{z^k z_j\} \cap \{x_1, \dots, x_t\} \neq \emptyset\}$ is finite, for each $j = 1, \dots, l$. Indeed, otherwise there exist $k_1 \neq k_2$ such that $z^{k_1 - k_2} = 1$, a contradiction. \square

Lemma 4. *If $x \in X$ is nontrivial, then $(x^m)^f$ is nontrivial.*

Proof. Suppose that there exists a nontrivial $x \in X$ such that $x^m \in \ker(f)$. Then for each $a \in A$ and each $u \in X$ we have

$$\begin{aligned} (a^{-mu} a^{mux^m})^f &= (a^{-mu})^f (a^{mux^m})^f = (a^{-mu})^f ((a^{mu})^f)^{(x^m)^f} \\ &= (a^{-mu})^f (a^{mu})^f = 1. \end{aligned}$$

Since $A^{m(x^m-1)} \leq \ker(f)$ and is normal in G , we have a contradiction. \square

Lemma 5. *The subgroup A^m is f -invariant.*

Proof. Let $a \in A$. Consider $T = \{c_1, \dots, c_r\}$ a transversal of A_0 in A , where r is a divisor of m . Since A^m is a subgroup of A_0 and $A = \bigoplus_{x \in X} B^x$, there exists $x_1, \dots, x_t, z_1, \dots, z_l \in X$ such that

$$\langle (c_i^m)^f \mid i = 1, \dots, r \rangle \leq B^{x_1} \oplus \dots \oplus B^{x_t}$$

and

$$\langle (a^m)^f \rangle \leq B^{z_1} \oplus \dots \oplus B^{z_l}.$$

Since $[G : H] = m$ and $Y^f \leq X$, it follows that $X^m \leq Y$. Fix a nontrivial $x \in X$ and let $z = (x^m)^f$.

For each integer k , define $i_k \in \{1, \dots, r\}$ such that

$$a^{x^{mk}} c_{i_k}^{-1} \in A_0.$$

Then

$$\left((a^{x^{mk}} c_{i_k}^{-1})^m \right)^f = \left((a^{x^{mk}} c_{i_k}^{-1})^f \right)^m \in A^m.$$

But $(a^{x^{mk}} c_{i_k}^{-1})^m = a^{mx^{mk}} c_{i_k}^{-m}$, thus

$$\left((a^{x^{mk}} c_{i_k}^{-1})^m \right)^f = \left(a^{mx^{mk}} \right)^f (c_{i_k}^{-m})^f = (a^{mf})^{z^k} c_{i_k}^{-mf}.$$

By Lemma 3, $z \neq 1$. There exists by Lemma 2 an integer k' such that

$$\{z^{k'} z_1, \dots, z^{k'} z_l\} \cap \{x_1, \dots, x_t\} = \emptyset,$$

and so,

$$(B^{z^{k'} z_1} \oplus \dots \oplus B^{z^{k'} z_l}) \cap (B^{x_1} \oplus \dots \oplus B^{x_t}) = 1.$$

It follows that

$$(a^{mf})^{z^{k'}} c_{i_k}^{-mf} \in A^m \cap [(B^{z^{k'} z_1} \oplus \dots \oplus B^{z^{k'} z_l}) \oplus (B^{x_1} \oplus \dots \oplus B^{x_t})].$$

But as

$$A^m = \bigoplus_{x \in X} B^{mx},$$

we conclude, $(a^{mf})^{z^{k'}} \in B^{mz^{k'} z_1} \oplus \dots \oplus B^{mz^{k'} z_l} \leq A^m$ and $a^{mf} \in A^m$. Hence, $(A^m)^f \leq A^m$. \square

With this last lemma, the proof of Theorem 2 is finished.

5. Example

Let L be a self-similar *abelian* group with respect to a simple triple (L, M, ϕ) ; then ϕ is a monomorphism. Define $B = \sum_{i \geq 1} L_i$, a direct sum of groups where $L_i = L$ for each i . Let X be cyclic group of order 2 and $G = BwrX$, the wreath product of B by X . Denote the normal closure of B in G by A ; then,

$$\begin{aligned} A &= B^X = \left(L_1 \oplus \sum_{i \geq 2} L_i \right) \times B \\ G &= A \bullet X. \end{aligned}$$

Define the subgroup of G

$$H = \left(M \oplus \sum_{i \geq 2} L_i \right) \times B ;$$

an element of H has the form

$$\beta = (\beta_1, \beta_2)$$

where

$$\begin{aligned} \beta_i &= (\beta_{ij})_{j \geq 1}, \beta_{ij} \in L \\ \beta_1 &= (\beta_{1j})_{j \geq 1}, \beta_{11} \in M. \end{aligned}$$

We note that $[G : H]$ is finite; indeed,

$$[A : H] = [L : M] \text{ and } [G : H] = 2[L : M].$$

Define the maps

$$\phi'_1 : M \oplus \left(\sum_{i \geq 2} L_i \right) \rightarrow B, \phi'_2 : B \rightarrow B$$

where for $\beta = (\beta_1, \beta_2) = ((\beta_{1j}), (\beta_{2j}))_{j \geq 1}, \beta_{11} \in M$,

$$\phi'_1 : \beta_1 \mapsto (\beta_{11}^\phi \beta_{12}, \beta_{13}, \dots), \phi'_2 : \beta_2 \mapsto (\beta_{22}, \beta_{21}, \beta_{23}, \dots).$$

Since L is abelian, ϕ'_1 is a homomorphism and clearly ϕ'_2 is a homomorphism as well.

Define the homomorphism

$$f : \left(M \oplus \sum_{i \geq 2} L_i \right) \times B \rightarrow A$$

by

$$f : (\beta_1, \beta_2) \mapsto ((\beta_1)^{\phi'_1}, (\beta_2)^{\phi'_2}).$$

Suppose by contradiction that K is a nontrivial subgroup of H , normal in G and f -invariant and let $\kappa = (\kappa_1, \kappa_2)$ be a nontrivial element of K . Since X permutes transitively the indices of κ_i , we conclude $\kappa_{i1} \in M$ for $i = 1, 2$. Let s_i (call it degree) be the maximum index of the nontrivial entries of κ_i ; if $\kappa_i = 0$ then write

$s_i = 0$. Choose κ with minimum $s_1 + s_2$; we may assume s_1 be minimum among those $s_i \neq 0$. Since

$$\begin{aligned}\kappa_1 &= (\kappa_{1j})_{j \geq 1}, \kappa_{11} \in M, \\ (\kappa_1)^{\phi'_1} &= (\kappa_{11}^\phi \kappa_{12}, \kappa_{13}, \dots),\end{aligned}$$

we conclude $(\kappa_1)^{\phi'_1}$ has smaller degree and therefore

$$\kappa_1 = (\kappa_{11}, e, e, e, \dots) \text{ or } (\kappa_{11}, \kappa_{11}^{-\phi}, e, e, \dots).$$

Suppose $\kappa_1 = (\kappa_{11}, e, e, e, \dots)$. As, $\kappa = (\kappa_1, \kappa_2) \in K$, we have $\kappa_{11} \in M$ and therefore

$$\begin{aligned}\kappa^f &= ((\kappa_1)^{\phi'_1}, (\kappa_2)^{\phi'_2}) \in K, \\ (\kappa_1)^{\phi'} &= ((\kappa_{11})^\phi, e, e, e, \dots), \\ (\kappa_{11})^\phi &\in M; \\ \kappa^{f^2} &= ((\kappa_1)^{(\phi'_1)^2}, \kappa_2), \\ (\kappa_1)^{(\phi'_1)^2} &= ((\kappa_{11})^{\phi^2}, e, e, e, \dots), \\ (\kappa_{11})^{\phi^2} &\in M; \text{ etc.}\end{aligned}$$

By simplicity of ϕ , this alternative is out. That is,

$$\kappa_1 = (\kappa_{11}, \kappa_{11}^{-\phi}, e, e, \dots), \kappa_{11} \in M.$$

Therefore

$$\begin{aligned}\kappa &= (\kappa_1, \kappa_2), \\ \kappa^x &= (\kappa_2, \kappa_1), \kappa^{xf} = ((\kappa_2)^{\phi'_1}, (\kappa_{11}^{-\phi}, \kappa_{11}, e, e, \dots)), \\ \kappa^{xfx} &= ((\kappa_{11}^{-\phi}, \kappa_{11}, e, e, \dots), (\kappa_2)^{\phi'_1})\end{aligned}$$

are elements of K and so, $\kappa_{11}^\phi \in M$. Furthermore,

$$\begin{aligned}\kappa^{xfxf} &= ((\kappa_{11}^{-\phi^2} \kappa_{11}, e, e, \dots), (\kappa_2)^{\phi'_1 \phi'_2}), \\ \kappa_{11}^{-\phi^2} \kappa_{11} &\in M;\end{aligned}$$

successive applications of f to κ^{xfx} produces $\kappa_{11}^{\phi^i} \in M$. Therefore, $\langle \kappa_{11}^{\phi^i} \mid i \geq 0 \rangle$ is a ϕ -invariant subgroup of M ; a contradiction is reached.

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